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On the semi-discretization of optimal control problems for networks of elastic strings: global optimality systems and domain decomposition ☆

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Abstract

We consider a star-graph as an examplary network, with elastic strings stretched along the edges. The network is allowed to perform out-of-the plane displacements. We consider such networks as being controlled at its simple nodes via Dirichlet conditions. The objective is to steer given initial data to final target data in a given time T with minimal control costs. This problem is discussed in the continuous as well as in the discrete case. We discuss an iterative domain decomposition technique and its discrete analogue. We prove convergence and show some numerical results.

Keywords: Domain decomposition; Networks of strings; Optimal control; Semi-discrete models

1. Introduction

Complex elastic multi-link structures can be considered as graphs $G(V,E)$ with edges E taken as structural elements like cables, strings, beams, plates, etc. At multiple nodes $V_M \subseteq V$ those structural elements are linked together via mechanical joints. While it is in principle possible to derive deterministic PDE-based models for very complex structures, handling such models is rather difficult from a numerical point of view. A compromise is achieved by some homogenization of such networks in parts where the structure is periodic. Nevertheless, even the reduced graphs obtained by such 'lumping' procedures can still be complicated, and a mathematical treatment along with its numerical realization is mandatory. See Lagnese et al. [9] for a survey.

The problem becomes particularly important when optimal control processes are exerted on such systems. We have devised and investigated dynamic domain decomposition techniques for such optimal control problems. The basic idea is to reduce such optimal control problems given on the

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entire network to analoguous ones restricted to the edges. This is done in an iteration such that the mechanical coupling histories are taken as loads on the substructure under consideration. The methods, which we describe below, are inherently parallel, see [8,11,14]. So far convergence results have been established on the infinite-dimensional level. In this paper we attempt to prove convergence for the semi-discrete problems.

Semi-discrete approximations to optimal control problems for parabolic equations (including control and state constraints) have been studied in Neittanmäki and Tiba [16]; see also the references therein. Fully discrete problems in this context have been considered in [5,18]. Semi-discrete hyperbolic problems have been considered in [19] and very recently in [7]. The results in [7] have been extended to graphs, like the ones discussed here in [4].

We emphazise that this paper is mainly concerned with the description and the anaylsis of the continous network model together with its semi-discrete approximations, and with the non-overlapping domain decomposition procedure both on the continuous and the semi-discrete level. Even though we give some numerical evidence in order to illustrate the method, we do not intend to give a detailed presentation of the numerical results.

As the notation is rather involved and the presentation of general networks is very complex and technical, we decided to fix ideas only in a very simple but typical network, namely, a star-graph (even with 3 branches, only). By this choice, which is made for simplicity of the presentation only, we exclude for instance networks with cycles. However, each network can be cut into star-graphs. Once domain decomposition is understood for such 'atoms', it can be applied to general graphs, also in 3-d-space. By this admittedly crude reduction we are able to write down the problem without any difficulty.

1.1. Problem formulation in the continuous case

Denote by $u_i(x,t),\ i=1,2,3,\ x\in(0,l),\ t\in(0,T)$, the vertical (out-of-plane) displacement of the elastic string i streched from the common joint at $x=0$ to the endpoints of the star.

We thus consider three coupled 1-d-wave equations (dots denote time derivatives, primes spatial derivatives)

$$\begin{aligned}
&\ddot{u}_i = u_i'' + F_i, \quad i=1,2,3,\ x\in(0,l),\ t\in(0,T),\\
&u_i(l,t) = f_i(t), \quad i=1,2,3,\ t\in(0,T),\\
&u_1(0,t) = u_2(0,t) = u_3(0,t), \quad t\in(0,T),\\
&\sum_{i=1}^{3} u_i'(0,t) = 0, \quad t\in(0,T),\\
&u_i(x,0) = \dot{u}_i(x,0) = 0, \quad x\in(0,l),\ i=1,2,3.
\end{aligned} \tag{1.1}$$

Eq. (1.1) represents the simplest nontrivial network problem for wave equations. Similar systems can be written down for parabolic, diffusion advection or Petrovski-type equations on graphs. See [10] for the general case of hyperbolic systems or Petrovski-systems, and von Below [20] for parabolic and related systems (see also [1,17]).

We consider spaces

$$H := \prod_{i=1}^{3} L^2(0,l),$$

$$V := \left\{ u \in \prod_{i=1}^{3} H^1(0,l) | u_1(0) = u_2(0) = u_3(0),\ u_1(l) = 0,\ i = 1,2,3 \right\}.$$

The problem of exact controllability can be put into the form:

$$\begin{cases} \text{For given target data } (u_T, \dot u_T) \in H \times V^\star, \\ \text{find controls } f_i \in L^2(0,T),\ i = 1,2,3 \text{ such that the } u_i \\ \text{satisfying (1.1) also satisfy the end condition} \\ u_i(x,T) = u_{iT}, \quad \dot u_i(x,T) = \dot u_{iT}, \quad x \in (0,l),\ i = 1,2,3. \end{cases} \tag{1.2}$$

Various variants for different systems, boundary and node-conditions have been investigated in [9].

Theorem 1. *Problem* (1.1), (1.2) *has a unique solution with minimal norm.*

Remark 2. Theorem 1 can be established by various methods, e.g. by moment theory, characteristics and multiplier techniques. It holds for arbitrary trees where at most one of the simple nodes is clamped and the other simple nodes are controlled. See [9,10].

In order to develop a sense which properties of the solutions of (1.1), (1.2) should hold for semi-discretizations, we give a

Proof of Theorem 1. We start with some energy inequalities. We define the total energy by

$$E(t) := \frac{1}{2} \left\{ \int_0^l \sum_{i=1}^{3} \dot u_i^2 + u_i'^2 \, dx \right\}. \tag{1.3}$$

Then for solutions u to (1.1) we have

$$\begin{aligned} \frac{dE(t)}{dt} &= \sum_{i=1}^{3} \int_0^l \{ \dot u_i \ddot u_i + u_i' \dot u_i' \} \, dx \\ &= \int_0^l \sum_{i=1}^{3} \dot u_i (u_i'' + F_i) \, dx \, dt + \sum_{i=1}^{3} u_i' \dot u_i |_0^l \, dt - \int_0^l \sum_{i=1}^{3} u_i'' \dot u_i \, dx \, dt \\ &= \sum_{i=1}^{3} \dot f_i(t) u_i'(l,t) + \sum_{i=1}^{3} \int_0^l F_i \dot u_i \, dx. \end{aligned} \tag{1.4}$$

If no inputs act on the system, we have conservation of energy.

Now, integrating (1.4) with respect to t gives

$$E(T) = E(0) + \sum_{i=1}^{3} \int_0^T \dot f_i u_i'(l,t) \, dt + \sum_{i=1}^{3} \int_0^T \int_0^l F_i \dot u_i \, dx \, dt.$$

In order to obtain energy estimates in terms of the inputs, we use standard energy multipliers $m_i(x)$ in order to obtain the general identity

$$\frac{1}{2}\int_0^T \sum_{i=1}^3 m_i(\dot u_i^2 + u_i'^2)|_0^l \,\mathrm{d}t = \int_0^l \sum_{i=1}^3 \dot u_i m_i u_i' \,\mathrm{d}x|_0^T - \int_0^T \int_0^l \sum_{i=1}^3 F_i m_i u_i' \,\mathrm{d}x\,\mathrm{d}t$$
$$+\frac{1}{2}\int_0^T \int_0^l \sum_{i=1}^3 m_i'(\dot u^2 + u'^2)\,\mathrm{d}x\,\mathrm{d}t \tag{1.5}$$

which relates the 'energy trace' with the total energy. Applying our particular boundary and transmission conditions and assuming that $m_i(x) = -1 + (2x/l)$ we obtain

$$\frac{1}{2}\int_0^T \sum_{i=1}^3 \{\dot u_i(0,t)^2 + u_i'(0,t)^2 + \dot u_i(l,t)^2 + u_i'(l,t)\}\,\mathrm{d}t$$
$$\leqslant C\left\{\int_0^T E(t)\,\mathrm{d}t + E(T) + E(0) + \int_0^T \int_0^l \sum_{i=1}^3 F_i^2 \,\mathrm{d}x\,\mathrm{d}t\right\}. \tag{1.6}$$

Inequality (1.6) which is referred to as a direct inequality establishes a so-called 'hidden regularity', as finite energy solutions have L^2-traces of the velocity and Neuman data, a fact, which is not directly seen from trace-theorems. On the other side by (1.4)

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leqslant \sum_{i=1}^3 \left\{\varepsilon u_i'(l,t)^2 + \frac{1}{4\varepsilon}\dot f_i(t)^2 + \varepsilon\int_0^l F_i(x,t)^2\,\mathrm{d}x + \frac{1}{4\varepsilon}\int_0^l \dot u_i(x,t)^2\,\mathrm{d}x\right\}$$

which upon integration and (1.6) gives

$$E(t) \leqslant E(0) + 2\varepsilon C\left\{\int_0^t E(s)\,\mathrm{d}s + E(t) + E(0) + \int_0^t \int_0^l \sum_{i=1}^3 F_i \,\mathrm{d}x\,\mathrm{d}t\right\}$$
$$+\varepsilon\int_0^t \int_0^l \sum_{i=1}^3 F_i(x,s)^2\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{4\varepsilon}\int_0^t \sum_{i=1}^3 \dot f_i(s)^2\,\mathrm{d}s$$
$$+\frac{1}{4\varepsilon}\int_0^t E(s)\,\mathrm{d}s. \tag{1.7}$$

Now, for ε sufficiently small ($2\varepsilon C < 1$) we absorb $E(t)$ appearing on the right-hand side of (1.7) into the left-hand side and apply Gronwall's inequality. We thus have

$$E(t) \leqslant C\left\{E(0) + \int_0^t \sum_{i=1}^3 \dot f_i^2(t)\,\mathrm{d}t + \int_0^t \int_0^l \sum_{i=1}^3 F_i^2\,\mathrm{d}x\,\mathrm{d}t\right\}. \tag{1.8}$$

Inequality (1.8) gives well-posedness of (1.1) for finite energy initial data and H^1-boundary data as well as L^2-distributed inputs. By transposition we obtain $H \times V^\star$-regularity of u with initial data $(u_0, u_1) \in H \times V^\star$, $F \in L^1(0,T,V^\star)$, $f \in L^2(0,T)^3$.

In Eq. (1.5) we may also take $m_i(x) = x$, and consider solutions ϕ_i to (1.1) with $f_i \equiv 0$, $F_i \equiv 0$, $i = 1,2,3$. We obtain

$$\frac{l}{2}\int_0^T \sum_{i=1}^3 \phi_i'(l,t)^2\,\mathrm{d}t = \int_0^t \sum_{i=1}^3 \dot\phi_i m_i \phi_i'\,\mathrm{d}x|_0^T + \int_0^T E_\phi(t)\,\mathrm{d}t \geqslant (T-2)E_\phi(0), \tag{1.9}$$

where we have used conservation of energy for ϕ. Defining the operator L_T

$$\begin{cases} L_T : \prod_{i=1}^{3} L^2(0,T) \to V^\star \times H \\ \qquad L_T f = (-\dot{u}(T), u(T)), \\ \text{where } u \text{ solves (1.1) in a weak sense,} \end{cases} \tag{1.10}$$

we observe that $L_T^\star$ is given by

$$L_T^\star : V \times H \to \prod_{i=1}^{3} L^2(0,T),$$
$$L_T^\star(\phi_0, \phi_1) = (\phi_i'(l, \cdot))_{i=1,2,3}. \tag{1.11}$$

Now, controllability (1.2) is equivalent to the surjectivity of L_T which is, in turn, equivalent to the condition

$$\|L_T^\star(\phi_0, \phi_1)\| \geqslant \gamma \|\phi_0, \phi_1\|_{V \times H}. \tag{1.12}$$

For $T > 2$, (1.9) is equivalent to (1.12). This proves Theorem 1. □

Obviously, the controls with minimum norm realizing the transfer described by (1.2) are given via the right-inverse of L_T:

$$f = L_T^\star (L_T L_T^\star)^{-1}(-z_1, z_0) := (-\dot{u}_T, u_T). \tag{1.13}$$

Thus, solving the symmetric problem

$$L_T L_T^\star(\phi_0, \phi_1) = (-z_1, z_0) \tag{1.14}$$

for the data $(\phi_0, \phi_1) \in V \times H$, $L_T^\star(\phi_0, \phi_1)$ is the solution of (1.1) with no other inputs than ϕ_0, ϕ_1 as final data — this is what is called a backwards running adjoint equation — with Neuman-traces at the controlled nodes. The optimality condition for this equality constrained quadratic optimization problem can be given as follows:

$$\begin{aligned} &\langle (p_T, \dot{p}_T), (-z_1, z_0) \rangle_{V \times H, V^\star \times H} = \int_0^T \sum_{i=1}^{3} |p_i'(l,t)|^2 \, \mathrm{d}t, \\ &f = L_T^\star(p_T, \dot{p}_T), \end{aligned} \tag{1.15}$$

where p solves the homogeneous problem (1.1) with $p(T) = p_T$, $\dot{p}(T) = \dot{p}_T$.

As is obvious from (1.13), the crucial property in order to succeed in solving (1.15) is to have lower bounds on $L_T L_T^\star$. Those are provided by (1.9), which are, therefore, called reverse inequalities. The procedure outline so far is essentially an application of the HUM-method of Lions, to wave equations on graphs. Even though we have exact controllability of the continuous model (1.1), it turns out that lower estimates like (1.9) do not hold uniformly in the spatial discretization parameter, once a 'classical' semidiscretization is applied. For a single string this fact has been pointed out by Infante and Zuazua [7]. For a network like (1.1) this is on going joint research with Brauer [4]. The reason for the lack of uniformity in the observability estimates (as (1.9)) obtained via 'classical' semi-discretization lies with the high-frequency behaviour of the semi-discrete approximations. This fact is due to the poor approximation of true eigenvalues by the eigenvalues of the finite-difference (finite-element) matrices. The effect on the numerical realization of exact controllability problems was first observed by Glowinski, see [5,6] for references.

The illposedness of the problem of exact-controllability for semi-discrete approximations can be cured by various methods:

- truncated SVD,
- Tychonov regularization,
- two-grid schemes,
- high frequency filtering.

The first method reduces to cutting down Fourier series expansions at high frequencies [7]. The second and third methods have been employed by Glowinski in various papers, see also [5,6].

From the point of view regularizing the equation $L_T L_T^\star(p_T, \dot p_T) = (-z_1, z_0)$, it is most natural to add $(1/k)I$, $k \gg 1$. There are other choices, taking into account different duality mappings between normed subspaces of energy spaces. It follows from standard theory that the regularized equation

$$
\begin{aligned}
&L_T L_T^\star(p_T, \dot p_T) + \frac{1}{k}(p_T, \dot p_T) = (-z_1, z_0),\\
&f = L_T^\star(p_T, \dot p_T) = (p_i'(l,\cdot))_{i=1,2,3}
\end{aligned}
\tag{1.16}
$$

is the optimality condition for the penalized optimal control problem

$$
\begin{aligned}
&\min_f \left\{ \frac{1}{2}\int_0^T \sum_{i=1}^3 f_i^2\,\mathrm{d}t + \frac{k}{2}\{\|u(T)-z_0\|_H^2 + \|\dot u(T) - z_1\|_{V^\star}^2\} =: J_k(f)\right\}\\
&u \text{ solves (1.1).}
\end{aligned}
\tag{1.17}
$$

We can show that (1.17) for $k \to \infty$ converges to (1.2) in a strong sense made precise below. The argument is similar to Lagnese and Leugering [8] for the Neumann case in $2-d$, or to Lagnese [11], and is hence omitted. We only indicate in which sense the transmission conditions hold in the limit. If we take the target data in $V \times H$, then the final data for p are $D(A) \times V$, $D(A) := \{u \in \prod_{i=1}^3 H^2(0,l) \cap V \mid \sum_{i=1}^3 u_i'(0) = 0\}$, provided $(u(T), \dot u(T)) \in V \times H$. However, with $D(A) \times V$-final data, p is in $C(0,T;D(A)) \cap C^1(0,T;V)$ having Neuman traces $p_i'(l,\cdot) \in H^1(0,T)$. Hence, because of $f_i = p_i'(l_j) \in H^1(0,T)$ the resulting u is in $C(0,T;V) \cap C^1(0,T,H)$, which closes the cycle. Thus, if we insist on $V \times H$-regularity for the target data, the transmission conditions are fulfilled in the classical L^2-sense. This is, of course, important also from a numerical point of view. We have

Theorem 3. *Let $(z_0, z_1) \in V \times H$. Then as $k \to \infty$ the solution $u(\cdot;k)$ of the optimality system*

$$
\begin{aligned}
&\ddot u_i = u_i'',\ \ddot p_i = p_i'', && i = 1,2,3,\ x \in (0,l),\ t \in (0,T),\\
&u_i(l,t) = p_i'(l,t),\ p_i(l,t) = 0, && i = 1,2,3,\ t \in (0,T),\\
&u_i(0,t) = u_j(0,t),\ p_i(0,t) = p_j(0,t), && i,j = 1,2,3,\ t \in (0,T),\\
&\sum_{i=1}^3 u_i'(0,t) = \sum_{j=1}^3 p_i'(0,t) = 0, && t \in (0,T),
\end{aligned}
\tag{1.18}
$$

$$u(\cdot,0) = 0 = \dot u(\cdot,0), \quad x \in (0,l),$$
$$p(\cdot,T) = kA^{-1}(\dot u(\cdot,T) - z_1), \tag{1.19}$$
$$\dot p(\cdot,T) = -k(u(\cdot,T) - z_0), \quad x \in (0,l)$$

satisfies

$$(u(\cdot;k), p(\cdot;k)) \to (u(\cdot), p(\cdot))$$

strongly in $C(0,T;H \times V) \cap C^1(0,T;V^\star \times H)$

$$p'(l_i;k) \to p'(l_i) \quad \text{strongly in } L^2(0,T)^3,$$

where $u(\cdot), p(\cdot)$ *satisfy* (1.18) *and*

$$u(\cdot,0) = \dot u(\cdot,0) = 0, \quad u(\cdot,T) = z_0, \quad \dot u(\cdot,T) = z_1, \quad p(\cdot,T) = p_T, \quad \dot p(\cdot,T) = \dot p_T, \tag{1.20}$$

and where $(p_T, \dot p_T) \in V \times H$ *is the unique solution of*

$$((p_T, \dot p_T), (-z_1, z_0)) = \int_0^T \sum_{i=1}^3 |p_i'(l,t)|^2 \,\mathrm{d}t. \tag{1.21}$$

The latter is the optimality system of the exact controllability problem.

Remark 4. As mentioned in the introduction, Theorems 1 and 3 can be extended to tree-like graphs for strings and Timoshenko beams; see [11].

2. Domain decomposition

For large networks the amount of work in solving optimality systems (1.18), (1.19), or (1.18), (1.20), (1.21) numerically is prohibitive. This is even more apparent when dealing with problems in higher dimensions as in [8]. It is, of course, always possible to discretize the problem first and then resort to some decomposition method of the system matrices. Our point of view, however, is that one should stay with the continuous model as long as possible, derive optimality conditions and control laws on that level and then discretize. By this method the physical properties of the substructures involved in a complex system are better represented. Indeed, it has been amply demonstrated by Benninghof and Boucher [3] that the opposite strategy can lead to disastrous results. There is another reason for discussing domain decomposition also on the continuous level. Namely, we are aiming at a modular device which can treat individual structural elements like strings, beams and also plates with individual solvers. Even though this does not appear plausible on the level of our model problem, we treat this case as an exemplaric situation. Domain decomposition algorithms for optimal control problems on graphs have been developed by the author using a basic idea which, in turn, goes back to Lions [15], and which has been applied to single-equation optimal control problems by Benamou [2]. We refer the reader to Leugering [12–14].

In this paper we discuss the basic version only.

2.1. The basic algorithm

We consider the following decoupling of the transmission conditions $(1.18)_{3,4}$:

$$
\begin{aligned}
-(u_i^{n+1})'(0,t)+\beta p_i^{n+1}(0,t) &= \beta\left(\frac{2}{3}\sum_{j=1}^{3} p_j^n(0,t)-p_i^n(0,t)\right)\\
&\quad+\frac{2}{3}\sum_{j=1}^{3}(u_j^n)'(0,t)-(u_i^n)'(0,t)=:\lambda_i^n,
\end{aligned}
$$

$$
\begin{aligned}
-(p_i^{n+1})'(0,t)-\beta u_i^{n+1}(0,t) &= -\beta\left(\frac{2}{3}\sum_{j=1}^{3}(u_j^n)(0,t)-u_i^n(0,t)\right)\\
&\quad+\frac{2}{3}\sum_{j=1}^{3}(p_j^n)'(0,t)-(p_i^n)'(0,t)=:-\mu_i^n.
\end{aligned}
\tag{2.1}
$$

If we delete the iteration indices $n, n+1$ and sum (2.1) over $i=1,2,3$ we see that $\sum_{i=1}^{3} u_i'(0,t)=\sum_{i=1}^{3} p_i'(0,t)=0$, and, using that information in (2.1) we find that u_i, p_i are continuous across $x=0$.

Now, (2.1) leads to a decoupling of (1.18) into a sequence of individual Dirichlet–Robin-type problems

$$\ddot{u}_i^{n+1}=(u_i^{n+1})'', \quad \ddot{p}_i^{n+1}=(p_i^{n+1})'', \quad x\in(0,l),\ t\in(0,T), \tag{2.2}$$

$$u_i^{n+1}(l,t)=(p_i^{n+1})'(l,t), \quad p_i^{n+1}(l,t)=0, \quad t\in(0,T), \tag{2.3}$$

$$-(u_i^{n+1})'(0,t)+\beta p_i^{n+1}(0,t)=\lambda_i^n, \quad t\in(0,T), \tag{2.4}$$

$$-(p_i^{n+1})'(0,t)-\beta u_i^{n+1}(0,t)=-\mu_i^n, \quad t\in(0,T), \tag{2.5}$$

$$u_i^{n+1}(x,0)=0=\dot{u}_i^{n+1}(x,0), \quad x\in(0,l), \tag{2.6}$$

for $i=1,2,3$.

While (2.6) is the 'decoupled' initial data for u, the final conditions for the adjoint problem in (1.19) include a global statement

$$p(\cdot,T)=kA^{-1}(\dot{u}(\cdot,T)-z_1),$$

involving the network operator A. Consequently we have to decompose the final condition, too. To this end consider the elliptic problem

$$
\begin{aligned}
&-y_i''+y_i=kg_i,\\
&y_i(l)=0, \quad y_i(0)=y_j(0)\quad \forall i,j=1,2,3, \quad \sum_{j=1}^{3} y_i'(0)=0,
\end{aligned}
\tag{2.7}
$$

with $g_i = A_i A^{-1}(\dot{u}(\cdot,T) - z_1)$, $A_i : H^1_{0-}(0,l) \to (H^1_{0-}(0,l))^\star$ the Riesz isomorphism with respect to the norm $|\cdot|_1$ ($H^1_{0-}(0,l) = \{\phi \in H^1(0,l) | \phi(l) = 0\}$).

The analogous decomposition procedure gives

$$\begin{aligned}
&-(y_i^{n+1})'' = kg_i,\\
&y_i^{n+1}(l) = 0, \quad -(y_i^{n+1})'(0) + \alpha y_i^{n+1}(0) = \tau_i^n,\\
&\tau_i^n := \alpha\left(\frac{2}{3}\sum_{j=1}^{3} y_j^{n\prime}(0) - y_j^{n\prime}(0)\right) - \left(\frac{2}{3}\sum_{j=1}^{3} y_j^{n\prime}(0) - y_j^{n\prime}(0)\right).
\end{aligned} \tag{2.8}$$

If we introduce the space $V_i = H^1_{0-}(0,l)$ endowed with the norm $\|\phi\|_{V_i} = (\int_0^l \phi'^2\,\mathrm{d}x + \alpha\phi(0))^{1/2}$, and $\mathscr{A}_i$ as its Riesz isomorphism to $V_i^\star$ we find for $p_i(0,T)$ solving (2.7)

$$\begin{aligned}
&p_i(\cdot,T) = k\mathscr{A}_i^{-1}\left(\dot{u}_i(\cdot,T) - z_1 + \frac{1}{k}\sigma_i\right),\\
&(\sigma_i,\phi)_{V_i^\star,V_i} := \tau_i\phi(0) \quad \forall\phi \in V_i.
\end{aligned} \tag{2.9}$$

This is because (2.9) is equivalent to

$$(p_i'(\cdot,T),\phi')_{L^2} + \alpha p_i(0,T)\phi(0) = k(\dot{u}_i(\cdot;T) - z_1,\phi)_{L^2} + \tau_i\phi(0), \tag{2.10}$$

which after integration by parts reduces to

$$-p_i'(0,T) + \alpha p_i(0,T) = \tau_i, \qquad -p_i'' = k(\dot{u}_i(\cdot,T) - z_1).$$

Therefore, the decomposition of the final value for the adjoint problem reads like

$$\begin{aligned}
&p_i^{n+1}(\cdot,T) = k\mathscr{A}^{-1}\left(\dot{u}_i^{n+1}(\cdot,T) - z_1 + \frac{1}{k}\sigma_i^n\right),\\
&\dot{p}(\cdot,T) = -k(u_i^{n+1}(\cdot,T) - z_0),\\
&(\sigma_i^n,\phi)_{V_i^\star,V_i} = \tau_i^n\phi(0) \quad \forall\phi \in V_i,\\
&\tau_i^n = \alpha\left(\frac{2}{3}\sum_{j=1}^{3} p_j^n(\cdot,T) - p_i^n(\cdot,T)\right)\\
&\qquad - \left(\frac{2}{3}\sum_{j=1}^{3} p_j^{n\prime}(\cdot,T) - p_j^{n\prime}(\cdot,T)\right).
\end{aligned} \tag{2.11}$$

The procedure is similar to the one in [8].

The local system then is given by Eqs. (2.2)–(2.6) and (2.11) with λ_i^n, μ_i^n given by (2.1).

As mentioned before Theorem 3, starting with smooth data, the decomposition procedure is well defined in the sense of traces. Convergence can then be shown as in [14].

Theorem 5. *Let $(u_T, \dot{u}_T) \in V \times H$. Then the procedure above converges to the solution of the global optimality system in $C(0,T; H \times V^\star)$.*

3. Semidiscretization

3.1. Discretization of the global problem

We introduce grid points on each of the three strings by $x_j{:=}jh,\ j=0:N+1,\ h{:=}l(N+1)$. We introduce the cell averages

$$u_j^i(t){:=}\frac{1}{h}\int_{x_j-h/2}^{x_j+h/2} u_i(x,t)\,\mathrm{d}x, \quad j=0:N+1,\ i=1,2,3, \tag{3.1}$$

over the cell $I_j{:=}(x_j-h/2, x_j+h/2)$, where u is properly extended on I_0, I_{N+1}. At the multiple node corresponding to $x=0$ we require

$$u_0^i(t)=z(t), \quad i=1,2,3,\ t\in(0,T). \tag{3.2}$$

At the controlled node we average (see also [19]) as follows:

$$u_{N+1}^i(t)+\rho(u_N^i(t)-u_{N+1}^i(t))=f_i(t), \quad i=1,2,3,\ t\in(0,T) \tag{3.3}$$

with $\rho<0$ to be specified below.

If we view (3.3) as

$$u_{N+1}^i(t)+h\rho\left(\frac{u_N^t(t)-u_{N+1}^i(t)}{h}\right)=f_i(t),$$

then (3.3) appears as Robin condition which tends to the Dirichlet condition for h small. On the other hand, the classical realization of the Dirichlet condition obviously corresponds to the choice $\rho=0$. The second-order approximation of (1.1) now reads

$$\begin{aligned}
&h^2\ddot{z}(t)=\sum_{i=1}^{3}u_1^i(t)-3z(t), \quad t\in(0,T),\\
&h^2\ddot{u}_j^i=\left(u_{j+1}^i-2u_j^i+u_{j-1}^i\right), \quad j=1:N,\ i=1,2,3,\ t\in(0,T),\\
&u_{N+1}^i+\rho(u_N^i-u_{N+1}^i)=f_i, \quad t\in(0,T),\\
&u_j^i(0)=u_j^{i,0}, \quad \dot{u}_j^i(0)=u_j^{i,1}, \quad t\in(0,T).
\end{aligned} \tag{3.4}$$

We denote by D_h the diagonal matrix with $D(1,1)=3,\ D(i,i)=2,\ i=2:3N+1$, that is the matrix carrying the edge-degrees of the nodes of the computational graph.

Furthermore, let A_h denote the vertex-to-vertex adjacency matrix corresponding to the computational graph. Then the so-called Laplacian $\mathscr{L}_h$ of that graph is defined by

$$\mathscr{L}_h{:=}D_h-A_h. \tag{3.5}$$

Upon introducing a global gridfunction $u_h=(z,u_1^1\dots u_N^1,u_1^2\dots u_N^2,u_1^3\dots u_N^3)$ and the classical realization of the Dirichlet conditions $u_{N+1}^i=0$, the homogeneous problem would be equivalent to

$$\ddot{u}_h+\frac{1}{h^2}\mathscr{L}_h u_h=0.$$

In this form, the discretization easily extends to general graphs. Note, in particular, that taking a rectangular lattice of strings, then after discretization as above, $\mathscr{L}_h$ is precisely the discrete Laplacian (i.e. five-point-star finite-difference, or the C^0-finite element discretization of the Laplace operator in $2-d$.) Indeed, it is this connection to higher dimensional problems, which also serves as a motivation to consider control problems on a $2-d$ $(3-d)$ irregular domain. If we incorporate the 'Dirichlet' conditions (3.3) into the matrix, we obtain

$$\begin{aligned}u_{N+1}^i&=\frac{1}{1-\rho}f_i-\frac{\rho}{1-\rho}u_N^i\\&=\frac{1}{1+|\rho|}f_i+\frac{|\rho|}{1+|\rho|}u_N^i,\end{aligned}$$

$$\begin{aligned}h^2\ddot{u}_N^i&=(u_{N+1}^i-2u_N^i+u_{N-1}^i)\\&=\left(-\left(2-\frac{|\rho|}{1+|\rho|}\right)u_N^i+u_{N-1}^i\right)+\frac{1}{1+|\rho|}f_i.\end{aligned}$$

Hence, with

$$\begin{aligned}T_h&:=\begin{pmatrix}2&-1&&\\-1&2&-1&\\&&\ddots&\\&&-1&2-|\rho|/1+|\rho|\end{pmatrix},\\v_h&:=(-1,0,\dots,0)^{\mathrm{T}},\quad c=-3.\end{aligned}\tag{3.6}$$

The overall stiffness matrix becomes

$$K_h=\frac{1}{h^2}\begin{pmatrix}c&v_h^{\mathrm{T}}&v_h^{\mathrm{T}}&v_h^{\mathrm{T}}\\v_h&T_h&&\\v_h&&T_h&\\v_h&&&T_h\end{pmatrix}.\tag{3.7}$$

Thus, with $F_h=(0,\dots,(1+|\rho|)^{-1}f_1,0\dots,(1+|\rho|)^{-1}f_2,\ 0\dots,((1+|\rho|)^{-1}f_3)^{\mathrm{T}}$ system (3.4) has the form

$$\begin{aligned}&\ddot{u}_h+K_hu_h=F_h,\quad t\in(0,T),\\&u_h(0)=u_h^0,\quad \dot{u}_h(0)=u_h^1.\end{aligned}\tag{3.8}$$

For a direct numerical processing of the initial-value problem (3.8) one would of course permute the coordinates in order to transform the block-arrow matrix (3.7) into a downwards pointing block arrow matrix, in order to reduce fill-in. For the analysis, however, we keep the notations as they stand.

Remark 6. A spectral analysis of $\mathscr{L}_h$ given by (3.5) can be found in [4]. A similar analysis applies to K_h as given by (3.7).

We proceed to establish energy estimates.

3.1.1. Conservation of energy

It is well-known that the classical semi-discretization of the $1-d$ wave equation with (classical) homogeneous Dirichlet conditions conserve the total energy. We wonder how the condition (3.3) affects this property.

We introduce $\rho_{ij}=1,\ j=1:N,\ \rho_{i0}=\frac{1}{3},\ i=1,2,3$ and

$$E_h(t):=\frac{h}{2}\left\{\sum_{i=1}^{3}\sum_{j=0}^{N}\rho_{ij}|\dot{u}^i_j(t)|^2+\left|\frac{u^i_{j+1}(t)-u^i_j(t)}{h}\right|^2\right\}. \tag{3.9}$$

Then we obtain by repeated summation by parts

$$\begin{aligned}\dot{E}_h&=h\sum_{i=1}^{3}\sum_{j=0}^{N}\left\{\rho_{ij}\ddot{u}^i_j\dot{u}^i_j+\frac{1}{h^2}(u^i_{j+1}-u^i_j)(\dot{u}^i_{j+1}-\dot{u}^i_j)\right\}\\&=\frac{2}{h}\sum_{i=1}^{3}(u^i_{N+1}-u^i_N)\dot{u}^i_{N+1}.\end{aligned} \tag{3.10}$$

If we use (3.3): $u^i_{N+1}-u^i_N=(u^i_{N+1}-f_i)/\rho$ then (3.10) reads like

$$\begin{aligned}\dot{E}_h&=\frac{1}{\rho h}\sum_{i=1}^{3}(u^i_{N+1}-f_i)\dot{u}^i_{N+1}\\&=\frac{1}{\rho h}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{3}(u^i_{N+1})^2-\frac{1}{\rho h}\sum_{i=1}^{3}f_i\dot{u}^i_{N+1}.\end{aligned} \tag{3.11}$$

Therefore, interpreting the first part as a contribution to the potential energy located at $x=l$ we have conservation of 'energy' for $f_i\equiv 0$, if we define

$$E^0_h=E_h+\frac{1}{2}\frac{1}{|\rho|h}\left(\sum_{i=1}^{3}u^i_{N+1}\right)^2 \quad (\rho<0),$$

i.e.,

$$\dot{E}^0_h(t)=0. \tag{3.12}$$

3.1.2. Energy multipliers

We consider the semi-discrete wave equation $(3.4)_2$ with an extra 'distributed' load F^i_j. We are looking for multiplier identities (1.5), (1.6) in the semi-discrete case,

$$\begin{aligned}\int_0^T\sum_{j=1}^{N}\left(\ddot{u}^i_j\cdot\frac{u^i_{j+1}-u^i_{j-1}}{2}\cdot j\right)\mathrm{d}t&=\int_0^T\sum_{j=1}^{N}F^i_j\cdot j\cdot\frac{u^i_{j+1}-u^i_{j-1}}{2}\,\mathrm{d}t\\&\quad+\frac{1}{h^2}\int_0^T\sum_{j=1}^{N}(u^i_{j+1}-2u^i_j+u^i_{j-1})j\frac{1}{2}(u^i_{j+1}-u^i_{j-1})\,\mathrm{d}t.\end{aligned} \tag{3.13}$$

Upon repeated summation by parts we obtain

$$
\begin{aligned}
&h\mathscr{X}_i(t)|_0^T + \frac{h}{2}\int_0^T \left\{ \sum_{i=1}^N \dot{u}^i_j \frac{\dot{u}_{j+1}+\dot{u}_{j-1}}{2} + \sum_{j=0}^N \left| \frac{u^i_{j+1}-u^i_j}{h} \right|^2 \right\} \mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^T \dot{u}^i_1 \dot{z}\,\mathrm{d}t - \int_0^T \frac{1}{h} z^2\,\mathrm{d}t \\
&= \frac{1}{2}\left(l-\frac{h}{2}\right)\int_0^T \dot{u}^i_N \dot{u}^i_{N+1}\,\mathrm{d}t + \frac{l}{h^2}\int_0^T (u^i_{N+1}-u^i_N)^2\,\mathrm{d}t \\
&\quad + \int_0^T \sum_{j=0}^n F^i_j h\, jh \frac{u^i_{j+1}-u^i_{j-1}}{2h}\,\mathrm{d}t.
\end{aligned} \tag{3.14}
$$

This is the basic semi-discrete energy multiplier-identity for a single string.

Now, by (3.3) we have $\dot{u}^i_N = (1/\rho)\dot{f} + (1-(1/\rho))\dot{u}_{N+1}$, and, therefore after some standard applications of properly scaled Cauchy–Schwartz estimates (3.14) can be written as

$$
\begin{aligned}
&l\int_0^T \sum_{i=1}^3 \left| \frac{u^i_{N+1}-u^i_N}{h} \right|^2 \mathrm{d}t + \left(1-\frac{1}{\rho}\right)\frac{1}{2}\left(l-\frac{h}{2}\right)\int_0^T \sum_{i=1}^3 |\dot{u}^i_{N+1}|^2\,\mathrm{d}t \\
&\leqslant Ch\int_0^T \sum_{i=1}^3 \sum_{j=1}^N \left\{ \rho_{ij}|\dot{u}^i_j|^2 + \left| \frac{u^i_{j+1}+u^i_j}{h} \right|^2 \right\} \mathrm{d}t \\
&\quad + h\sum_{i=1}^3 \mathscr{X}_i(t)\bigg|_0^T + \int_0^T \sum_{i=1}^3 \sum_{j=1}^N hF^i_j\, jh \frac{u^i_{j+1}-u^i_{j-1}}{2h}\,\mathrm{d}t \\
&\quad + \frac{1}{4\rho\varepsilon_2}\left(l-\frac{h}{2}\right)\int_0^T \sum_{i=1}^3 |\dot{f}_i|^2\,\mathrm{d}t + \frac{\varepsilon_2}{4\rho}\left(l-\frac{h}{2}\right)\sum_{i=1}^3 \int_0^T |\dot{u}^i_{N+1}|^2\,\mathrm{d}t \\
&\quad + \frac{h}{2}\frac{\varepsilon_1}{4}\sum_{i=1}^3 \int_0^T |\dot{u}^i_{N+1}|^2\,\mathrm{d}t,
\end{aligned} \tag{3.15}
$$

with some positive ε_i. On the other hand, integrating (3.11) with respect to t and applying a suitable Cauchy–Schwarz estimate gives

$$
E_h(T) = E_h(0) + \frac{1}{2\varepsilon_3}\int_0^T \sum_{i=1}^3 \left| \frac{u^i_{N+1}-u^i_N}{h} \right|^2 \mathrm{d}t + \frac{\varepsilon_3}{2}\int_0^T \sum_{i=1}^3 |\dot{u}^i_{N+1}|^2\,\mathrm{d}t, \tag{3.16}
$$

again for some positive ε_3. We wish to absorb boundary terms into the left-hand side of (3.15). To this end we estimate

$$
\begin{aligned}
h\sum_{i=1}^3 \mathscr{X}_i(t) &= \sum_{i=1}^3 \sum_{j=1}^N h\dot{u}^i_j\, jh \frac{u^i_{j+1}-u^i_{j-1}}{2h} \\
&\leqslant h\sum_{i=1}^3 \sum_{j=1}^N \frac{1}{2}|\dot{u}^i_j|^2 + \frac{1}{h^2}\frac{l}{4}(|u^i_{j+1}-u^i_j|^2 + |u^i_j - u^i_{j-1}|^2)
\end{aligned}
$$

$$\leqslant h\sum_{i=1}^{\varepsilon}\sum_{j=0}^{N}\frac{1}{2}|\dot{u}_j^i|^2+\frac{h}{2}\left|\frac{u_{j+1}^i-u_j^i}{h}\right|^2$$
$$\leqslant \max(1,l)E_h(t)=:C_2E_h(t).$$

Hence

$$h\sum_{i=1}^{3}\mathscr{X}_i(t)\Big|_0^T\leqslant C_3Eh(0)+\frac{1}{2\varepsilon_3}\int_0^T\sum_{i=1}^{3}\left|\frac{u_{N+1}^i-u_N^i}{h}\right|^2\mathrm{d}t+\frac{\varepsilon_3}{2}\int_0^T\sum_{i=1}^{3}|\dot{u}_{N+1}^i|^2\,\mathrm{d}t. \tag{3.17}$$

We have for $\varepsilon_2,\varepsilon_1,\varepsilon_3-l/2$ small, and $|\rho|\in(0,1/2)$

$$l-\frac{1}{2\varepsilon_3}>0,\quad \left(1+\frac{1}{|\rho|}\right)\frac{1}{2}\left(l-\frac{h}{2}\right)-\frac{\varepsilon_2}{4\rho}\left(l-\frac{h}{2}\right)-\frac{h\varepsilon_1}{24}-\frac{\varepsilon_3}{2}>0.$$

Therefore, we obtain from (3.15) and (3.17)

$$\int_0^T\sum_{i=1}^{3}\left|\frac{u_{N+1}^i-u_N^i}{h}\right|^2\mathrm{d}t+\int_0^T\sum_{i=1}^{3}|\dot{u}_{N+1}^i|^2\,\mathrm{d}t$$
$$\leqslant C\rho\left\{\int_0^T E_h(t)\,\mathrm{d}t+E_h(0)+\sum_{i=1}^{3}\sum_{j=0}^{N}\int_0^T|F_j^i|^2h\,\mathrm{d}t+\sum_{i=1}^{3}\int_0^T|\dot{f}_i|^2\,\mathrm{d}t\right\}\mathrm{d}t. \tag{3.18}$$

This is a discrete analogue of (1.7).

Note that as $\rho\to 0$ inequality (3.18) becomes obsolete. That is for $\rho=0$, which is the classical realization of Dirichlet inputs, we do not obtain an inequality like (3.18). Using (3.16) and (3.18) we finally obtain with a generic constant C and $T\geqslant 0$

$$E_h(T)\leqslant C\left\{\int_0^T E_h(t)\,\mathrm{d}t+E_h(0)+\sum_{i=1}^{3}\sum_{j=1}^{N}\int_0^T|F_j^i|^2h\,\mathrm{d}t+\sum_{i=1}^{3}\int_0^T|\dot{f}_i|^2\,\mathrm{d}t\right\}. \tag{3.19}$$

Hence, applying Gronwall's inequality we obtain

$$\max_{t\in[0,T]}E_h(t)\leqslant C\left\{E_h(0)+\sum_{i=1}^{3}\sum_{j=1}^{N}\int_0^T|F_j^i|^2h\,\mathrm{d}t+\sum_{i=1}^{3}\int_0^T|\dot{f}_i|^2\,\mathrm{d}t\right\}. \tag{3.20}$$

And finally we use (3.20) in (3.18) to obtain

$$\int_0^T\sum_{i=1}^{3}\left|\frac{u_{N+1}^i-u_N^i}{h}\right|^2\mathrm{d}t+\int_0^T\sum_{i=1}^{3}|\dot{u}_{N+1}^i|^2\,\mathrm{d}t$$
$$\leqslant\tilde{C}\left\{E_h(0)+\sum_{i=1}^{3}\sum_{j=1}^{N}\int_0^T|F_j^i|^2h\,\mathrm{d}t+\sum_{i=1}^{3}\int_0^T|\dot{f}_i|^2\,\mathrm{d}t\right\}. \tag{3.21}$$

Inequalities (3.20), (3.21) are crucial in that they show well-posedness and regularity of the semi-discrete system.

We now consider the discrete control-to-state-map, i.e., the semi-discrete analogue of (1.10). We need to work in the original continuous energy spaces. We use piecewise constant extensions of gridfunctions $(\varphi)_{j=0}^N$,

$$P_h(\varphi)_{j=0}^N = \begin{cases} z, & x \in (0, l/2), \\ \varphi_j^i, & x \in I_j,\ j = 1:N,\ i = 1,2,3, \\ 0, & x \in (0, h/2) \end{cases} \tag{3.22}$$

and define

$$\begin{aligned} &u_h^i := P_h(u_j^i(T))_{j=0}^N, \quad \dot{u}_h^i = P_h(\dot{u}_j^i(T))_{j=0}^N, \\ &L_{T,h} f := (-\dot{u}_h, u_h). \end{aligned} \tag{3.23}$$

Then

$$L_{T,h}^\star(\phi^0, \phi^1) = \left(\frac{\phi_1^i - \phi_0^i}{h} \right)_{i=1,2,3} \in L^2(0,T)^3. \tag{3.24}$$

Alluding to (1.13), (1.14) we want to minimize

$$\inf_{f \in \mathscr{R}(L_{T,h}^\star)} \|L_{T,h} f - (-z_1, z_0)\|_{V^\star \times H} \tag{3.25}$$

subject to an a priori bound

$$\|f\|_{L^2} \leqslant \frac{1}{\gamma} \|(-z_1, z_0)\|_{V^\star \times H}, \tag{3.26}$$

where $\gamma > 0$ is the lower bound in (1.12). It is important to note that $\|L_{T,h}^\star \phi\|$ is not uniformly bounded below in terms of the discretization parameter h, i.e., for each h we have a lower bound γ_h such that $\gamma_h \to 0$ as $h \to 0$. This has been shown in [4] for the standard semi-discretization. Note that (3.25), (3.26) has a unique minimizer.

We proceed to show strong pointwise convergence of $L_{T,h}$, $L_{T,h}^\star$ as h tends to 0. Let $(u_j^i)_{\substack{i=1,2,3\\ j=0,\dots,N+1}}$ be the solution of (3.4) with zero initial data, and $U_j^i(t) := \int_0^t u_j^i(s)\,\mathrm{d}s$. Then (U_j^i) solves the same system with $g_i(t) := \int_0^t f_i(s)\,\mathrm{d}s$. By (3.20) we have

$$E_h(U;T) \leqslant C \sum_{i=1}^3 \int_0^T |f_i|^2\,\mathrm{d}t.$$

Now,

$$\|L_{T,h} f\|_{V^\star \times H}^2 = \|(-\dot{u}_h, u_h)\|_{V^\star \times H}^2 \leqslant C E_h(U;T),$$

and, hence,

$$\|L_{T,h} f\|_{V^\star \times H}^2 \leqslant c \sum_{i=1}^3 \int_0^T |f_i|^2\,\mathrm{d}t. \tag{3.27}$$

Thus, $\{L_{T,h}\}$, $\{L_{T,h}^\star\}$ are uniformly bounded with respect to h. From this point on the arguments given in [19] apply to the global system. The idea is to take controls in $C_0^\infty(0,T)$ to the extent that the corresponding solutions u^i, are C^∞, too. Then, the difference

$$\mathscr{R}_j^i(t) := \int_0^t r_j^i(s)\,\mathrm{d}s := \int_0^t (u^i(x_j, s) - u_j^i(s))\,\mathrm{d}s$$

satisfies the inhomogeneous equation with

$$F^i_j(t;h) = \int_0^t (u^i)''(x_j,s) - \frac{1}{h^2}(u^i(x_{j+1},s) - 2u^i(x_j,s) + u^i(x_{j-1},s))\,\mathrm{d}s,$$

$$g_i(t;h) = \sigma \int_0^t (u^i(l-h,s) - u^i(l,s))\,\mathrm{d}s,$$

and zero initial conditions. As h tends to zero,

$$\sum_{i=1}^{3}\sum_{j=1}^{N}\int_0^T F^i_j(t;h)h\,\mathrm{d}t \to 0,$$

$$\sum_{i=1}^{3}\int_0^T g_i(t;h)\,\mathrm{d}t \to 0.$$

Hence $E_h(R;T) \to 0$, as $h \to 0$. This implies

$$\|L_{h,T}f - L_T f\| \to 0 \quad \text{as } h \to 0,$$

for f in $C_0^\infty(0,T)^2$. Similarily

$$\|L^\star_{h,T}z - L^\star_T z\| \to 0 \quad \text{as } h \to 0$$

for smooth final data z. Using density arguments and the Banach–Steinhaus theorem as in [19] we conclude

Theorem 7. *The family of operators $L_{T,h}, L^\star_{T,h}$ strongly converge pointwise to $L_T, L^\star_T$ as h tends to zero.*

Theorem 8. *Let $T > 2l$ and let f_h be the solution to the minimization problem* (3.25), *then $\|L_{T,h}f_h - (-z_1,z_0)\|_{H\times V^\star} \to 0$, as $h \to 0$.*

The proof of Theorem 8 is exactly as in [19]. Now, obviously,

$$\begin{aligned}|(f_h, L^\star_T c) - (z,c)| &\leqslant |(f_h, L^\star_T c) - (L^\star_{T,h}c) + |(f_h, L^\star_{T,h}c) - (z,c)| \\ &\leqslant \|f_h\|\,\|(L^\star_T - L^\star_{T,H})c\| + \|L_{T,h}f_h - z\|\,\|c\|.\end{aligned}$$

Hence, Theorems 7 and 8 give some weak convergence of the controls f_h of the semi-discrete control problems if we impose the a priori norm bound (3.26):

Theorem 9. *Let $T > 2l$ and let f_h be the solution to* (3.25), (3.26). *Then for $v = L^\star_T c$, $c \in V \times H$ we have*

$$(f_h, v)_{L^2(0,T)^3} \to (z,c)_{V^\star\times H, V\times H}$$

as $h \to 0$.

3.2. Semi-discretization of the global optimal control problem with final-state penalization

Instead of minimizing the distance between the final states and given target states over controls in the range of the adjoint control-to-state operator, as in (3.25), we now seek to minimize the control cost and the derivation from the given target simultaneously. This amounts to the following semi-discrete optimal control problem which is the semi-discrete counterpart of (1.17) (with weights $v_j^i, w_j^i > 0$),

$$\min_f \left\{ J_{h,k}(f) := \frac{v}{2} \sum_{i=1}^{3} \int_0^T f_i^2 \,\mathrm{d}t + \frac{k}{2} \left\{ \sum_{i=1}^{3} \sum_{j=0}^{N} (u_j^i(T) - z_j^i)^2 w_j^i + \sum_{i=1}^{3} h \sum_{j=0}^{N} (\dot{u}_j^i(T) - \dot{z}_j^i)^2 v_j^i \right\} \right\} \tag{3.28}$$

subject to (3.4).

It is sufficient to concentrate on zero initial data in (3.4). Let y be a solution of (3.4) with controls g_i instead of f_i. Then the directional derivative of $J_{h,k}(f)$ at g is easily computed and, hence, the optimality condition reads

$$v \sum_{i=1}^{3} \int_0^T f_i g_i \,\mathrm{d}t + k \sum_{i=1}^{3} h \sum_{j=0}^{N} (u_j^i(T) - z_j^i) y_j^i(T) w_j^i + k \sum_{i=1}^{3} h \sum_{j=0}^{N} (\dot{u}_j(T) - \dot{z}_j^i) \dot{y}_j^i(T) v_j^i = 0 \tag{3.29}$$

$\forall g_i \in L^2(0,T),\ i = 1,2,3.$

Now let p solve (3.4) with homogeneous boundary conditions and with final conditions (rather than initial conditions)

$$\begin{aligned} p_j^i(T) &= k(\dot{u}_j^i(T) - \dot{z}_j^i) v_j^i, \\ \dot{p}_j^i(T) &= -k(u_j^i(T) - z_j^i) w_j^i. \end{aligned} \tag{3.30}$$

We multiply (3.4) for p_j^i with y_j^i and integrate with respect to time. Applying summation by parts we obtain

$$\begin{aligned} 0 = &-kh \sum_{i=1}^{3} \sum_{j=1}^{N} [(\dot{u}_j^i(T) - \dot{z}_j^i) v_j^i y_j^i(T) + (u_j^i(T) - z_j^i) w_j^i y_j^i(T)] \\ &- \frac{1}{h} \int_0^T p_0^0 \sum_{i=1}^{3} y_1^i \,\mathrm{d}t + \frac{1}{h} \int_0^T y_0^0 \sum_{i=1}^{3} p_1^i \,\mathrm{d}t \\ &+ \frac{1}{h} \int_0^T \sum_{i=1}^{3} (p_N^i y_{N+1}^i - p_{N+1}^i y_N^i) \,\mathrm{d}t, \end{aligned} \tag{3.31}$$

with $p_0^0 = p_0^i,\ i = 1,2,3$ ($y_0^0,\ u_0^0$ analogous).

Also

$$\begin{aligned} 0 &= h \int_0^T \left[\ddot{p}_0^0 - \frac{1}{h^2} \left(\sum_{i=1}^{3} p_1^i - 3p_0^0 \right) \right] y_0^0 \,\mathrm{d}t \\ &= h(\dot{p}_0^0(T) y_0^0(T) - p_0^0(T) \dot{y}_0^0(T)) + h \left[p_0^0 \ddot{y}_0^0 - \frac{1}{h^2} \left(\sum_{i=1}^{3} p_1^i - 3p_0^0 \right) y_0^0 \right] \mathrm{d}t. \end{aligned} \tag{3.32}$$

But

$$\frac{1}{h}\int_0^T p_0^0 \sum_{i=1}^{3} y_1^i \,\mathrm{d}t = h\int_0^T p_0^0\left(\ddot{y}_0^0 + 3y_0^0\frac{1}{h^2}\right)\mathrm{d}t.$$

Therefore, adding (3.31) and (3.32) gives

$$\frac{1}{h}\int_0^T \sum_{i=1}^{3}(p_N^i y_{N+1}^i - p_{N+1}^i y_N^i)\,\mathrm{d}t = k\sum_{i=1}^{3} h\sum_{j=0}^{N}\{(u_j^i(T)-z_j^i)y_j^i(T)w_j^i + (\dot{u}_j^i(T)-\dot{z}_j^i)\dot{y}(T)v_j^i\}. \tag{3.33}$$

The boundary conditions read as

$$\begin{aligned} &p_{N+1}^i + \rho(p_N^i - p_{n+1}^i) = 0,\\ &y_{N+1}^i + \rho(y_N^i - y_{N+1}^i) = g_i, \quad i = 1,2,3. \end{aligned} \tag{3.34}$$

Hence,

$$\begin{aligned} p_{N+1}^i &= -\frac{\rho}{1-\rho}p_N^i = \frac{|\rho|}{1+|\rho|}p_N^i,\\ y_{N+1}^i &= \frac{1}{1+|\rho|}g_i + \frac{|\rho|}{1+|\rho|}y_N^i. \end{aligned}$$

Therefore, the directional derivative $J_{h,k}(f)(g)$ can be written as

$$v\sum_{i=1}^{3}\int_0^T f_i g_i\,\mathrm{d}t + \frac{1}{h}\sum_{i=1}^{3}\int_0^T \frac{1}{1+|\rho|}p_N^i g_i\,\mathrm{d}t = \delta J_{h,k}(f)(g) \tag{3.35}$$

$\forall g_i \in L^2(0,T)$. From (3.35) it follows that the optimality condition (3.29) reduces to

$$f_i(t) = -\frac{1}{v}\frac{1}{1+|\rho|}\frac{p_N^i(t)}{h}, \quad i = 1,2,3,\ t\in(0,T). \tag{3.36}$$

Eq. (3.36) can also be written as

$$f_i(t) = \frac{1}{v}\frac{p_N^i(t) - p_{N+1}^i(t)}{h}, \quad i = 1,2,3,\ t\in(0,T). \tag{3.37}$$

It is clear that for $\rho = 0$ (the classical discretization) we have to resort to (3.36). Now, for $h \to 0$ (3.37) tends to $f_i = (1/v)(p^i)'(l_i)$ as in (1.16) (normal derivative).

We conclude that solving the optimal control problem (3.28) amounts to solving the equation

$$L_{T,h}L_{T,h}^{\star}(p_h, \dot{p}_h) + \frac{1}{k}(p_h, \dot{p}_h) = (-\dot{z}_h, z_h), \tag{3.38}$$

with the notation of (3.23). Let $(p_h, \dot{p}_h)$ be the unique solution to (3.38). We have $(-\dot{z}_h, z_h) \to (-\dot{z}, z)_{V^\star\times H}$. Moreover

$$\begin{aligned} &\left((p_h, \dot{p}_h), \left(L_T L_T^{\star} + \frac{1}{k}I\right)(r,\dot{r})\right) - ((-\dot{z}_h, z_h), (r_j, \dot{r}))\\ &\quad \leqslant \|(p_h, \dot{p}_h)\|\,\|(L_{T,h}L_{T,h}^{\star} - L_T L_T^{\star})(r,\dot{r})\| \end{aligned}$$

$$+\left\|\left(L_{T,h}L^{\star}_{T,h}+\frac{1}{k}I\right)(p_h,\dot{p}_h)-(-\dot{z}_h,z_h)\right\|\,\|(r,\dot{r})\|$$
$$\to 0,\quad \text{as } h\to 0. \tag{3.39}$$

We thus have weak convergence of $(p_h,\dot{p}_h)$ and, hence, we also have weak convergence of the controls.

Theorem 10. *The controls* $f_h=L^{\star}_{T,h}(p_h,\dot{p}_h)$ *converge weakly to the controls* f *of the optimal control problem* (1.17).

This result is independent of k. And, indeed, for $k\to\infty$ we are back to Theorem 9. If k is kept fixed, we know that $(L_{T,h}L^{\star}_{T,h}+(1/k)I)$ is invertible with uniformly bounded inverse. In addition, since $L_TL^{\star}_T+(1/k)$ is also invertible, we have, because of the strong convergence of $L_{T,h},L^{\star}_{T,h}$ to $L_T,L^{\star}_T$, strong convergence of f_n as $h>0$.

Theorem 11. *For each fixed* $k>0$, *the controls* f_h *converge strongly to* f, *as* $h\to 0$.

3.3. Semi-discretization of the domain decomposition iteration

We consider the semi-discrete counterpart of (2.1)–(2.6). For $n\geqslant 0$

$$\ddot{u}^{i,n+1}_j=\frac{1}{h^2}(u^{i,n+1}_{j-1}-2u^{i,n+1}_j+u^{i,n+1}_{j+1}),\quad i=1,2,3,\ j=1:N,\ t\in(0,T), \tag{3.40}$$

$$\ddot{p}^{i,n+1}_j=\frac{1}{h^2}(p^{i,n+1}_{j-1}-2p^{i,n+1}_j+p^{i,n+1}_{j+1}),\quad i=1,2,3,\ j=1:N,\ t\in(0,T), \tag{3.41}$$

$$u^{i,n+1}_{N+1}+\rho(u^{i,n+1}_N-u^{i,n+1}_{N+1})=f_i,\quad i=1,2,3,\ t\in(0,T), \tag{3.42}$$

$$p^{i,n+1}_{N+1}+\rho(p^{i,n+1}_N-p^{i,n+1}_{N+1})=0,\quad i=1,2,3,\ t\in(0,T), \tag{3.43}$$

$$\begin{aligned}-\frac{1}{h}(u^{i,n+1}_1-u^{i,n+1}_0)+\beta p^{i,n+1}_0&=\beta\left(\frac{2}{3}\sum_{k=1}^{3}p^{k,n}_0-p^{i,n}_0\right)\\&\quad+\frac{1}{h}\left[\frac{2}{3}\sum_{k=1}^{3}(u^{k,n}_1-u^{k,n}_0)-(u^{i,n}_1-u^{i,n}_0)\right],\quad i=1,2,3,\ t\in(0,T),\end{aligned} \tag{3.44}$$

$$\begin{aligned}-\frac{1}{h}(p^{i,n+1}_1-p^{i,n+1}_0)-\beta u^{i,n+1}_0&=-\beta\left(\frac{2}{3}\sum_{k=1}^{3}u^{k,n}_0-u^{i,n}_0\right)\\&\quad+\frac{1}{h}\left[\frac{2}{3}\sum_{k=1}^{3}(p^{k,n}_1-p^{k,n}_0)-(p^{k,n}_1-p^{i,n}_0)\right],\quad i=1,2,3,\ t\in(0,T),\end{aligned} \tag{3.45}$$

$$u_j^{i,n+1}(0) = \dot{u}_j^{i,n+1}(0) = 0, \quad i = 1,2,3, \; j = 0 : N+1, \tag{3.46}$$

$$p_j^i(T) = k(\dot{u}_j^i(T) - \dot{z}_j^i)v_j^i, \quad i = 1,2,3, \; j = 0 : N+1, \tag{3.47}$$

$$\dot{p}_j^i(T) = -k(u_j^i(T) - z_j^i)w_j^i, \quad i = 1,2,3, \; j = 0 : N+1.$$

If $u_0^i = u_0^j, \; i,j = 1,2,3,$ then

$$u_0^i = \frac{2}{3}\sum_{k=1}^{3} u_0^k - u_0^i.$$

Therefore, if we drop the iteration index n in (3.45) and if we assume

$$\sum_{k=1}^{3} \frac{p_1^k - p_0^k}{h} = 0, \tag{3.48}$$

we obtain (3.45) ((3.44)). On the other hand, if we assume (3.44), (3.45) to hold with the iteration indices $n, n+1$ dropped, then $u_0^i = u_0^j \; i, \; j = 1,2,3, \; t \in (0,T)$ and (3.48) ($t \in (0,T)$) will follow. It is apparent that in the limit, as $h \to 0$, the transmission conditions (1.18), (1.19) of the continuous model are satisfied. Hence, the method is consistent with the continuous model. We can also employ second order accurate oblique difference approximations to the normal derivatives.

It is also apparent that (3.40)–(3.47) is not, on the discrete level, a domain decomposition method consistent with the semi-discrete global problem: we do not have dynamics at $j = 0$! A method that is consistent on the h-level with (3.4) and which contains an inertia term at $j = 0$ is currently under investigation.

As for a proof of convergence of the iterative process, we consider the error terms

$$\begin{aligned} \tilde{u}_j^{i,n+1} &:= u_j^{i,n+1} - u_j^i, \\ \tilde{p}_j^{i,n+1} &:= p_j^{i,n+1} - p_j^i, \end{aligned} \tag{3.49}$$

where $u_j^{i,n+1}, p_j^{i,n+1}$ solve (3.40)–(3.47) and u_j^i, p_j^i solve (3.40)–(3.43), (3.46), (3.47) without the index $n+1$, and

$$\begin{aligned} &u_0^i = u_0^j, \quad p_0^i = p_0^j, \quad i = 1,2,3, \; t \in (0,T), \\ &\sum_{k=1}^{3} \frac{u_1^k - u_0^k}{h} = \sum_{k=1}^{3} \frac{p_1^k - p_0^k}{h} = 0, \quad t \in (0,T). \end{aligned} \tag{3.50}$$

The errors satisfy (3.40)–(3.47) with

$$\tilde{p}_j^i(T) = k\dot{u}_j^i(T)v_j^i, \quad \dot{\tilde{p}}_j^i(T) = -ku_j^i(T)w_j^i, \; i = 1,2,3, \; j = 0 : N+1. \tag{3.51}$$

Taking squares on both sides of (3.44), (3.45) we obtain

$$\frac{1}{h^2}(\tilde{u}_1^{i,n+1} - \tilde{u}_0^{i,n+1})^2 + \beta^2(\tilde{p}_0^{i,n+1})^2 - 2\frac{\beta}{h}\tilde{p}_0^{i,n+1}(\tilde{u}_1^{i,n+1} - \tilde{u}_0^{i,n+1})$$

$$= \beta^2\left(\frac{4}{9}\sum_{k=1}^{3}\tilde{p}_0^{k,n}\right)^2 - 2\beta\frac{2}{3}\sum_{k=1}^{3}\tilde{p}_0^{k,n}\tilde{p}_0^{k,n} + \beta^2(\tilde{p}_0^{i,n})^2$$

$$+\frac{1}{h^2}\left[\frac{4}{9}\left(\sum_{k=1}^{3}(\tilde{u}_1^{k,n}-\tilde{u}_0^{k,n})\right)^2-\frac{4}{3}\sum_{k=1}^{3}(\tilde{u}_1^{k,n}-\tilde{u}_0^{k,n})(\tilde{u}_1^{i,n}-\tilde{u}_0^{i,n})+(\tilde{u}_1^{i,n}-\tilde{u}_0^{i,n})^2\right]$$

$$+2\beta\left(\frac{2}{3}\sum_{k=1}^{3}\tilde{p}_0^{k,n}-\tilde{p}_0^{i,n}\right)\frac{1}{h}\left[\frac{2}{3}\sum_{k=1}^{3}(\tilde{u}_1^{k,n}-\tilde{u}_0^{k,n})-\tilde{u}_1^{i,n}-\tilde{u}_1^{i,n})\right]. \tag{3.52}$$

We sum (3.52) over $i=1,2,3$

$$\sum_{i=1}^{3}\left|\frac{1}{h}(\tilde{u}_1^{i,n+1}-\tilde{u}_0^{i,n+1})\right|^2+\beta^2|\tilde{p}_0^{i,n+1}|^2+2\beta\sum_{i=1}^{3}\tilde{p}_0^{i,n+1}\frac{1}{h}(\tilde{u}_1^{i,n+1}-\tilde{u}_0^{i,n+1})$$

$$=\sum_{i=1}^{3}\left|\frac{1}{h}(\tilde{u}_i^{i,n}-\tilde{u}_0^{i,n})\right|^2+\beta^2|\tilde{p}_0^{i,n}|^2+2\beta\sum_{i=1}^{3}\left|\tilde{p}_0^{i,n}\frac{1}{h}(\tilde{u}_1^{i,n}-\tilde{u}_0^{i,n})\right|. \tag{3.53}$$

The same procedure applied to (3.45) gives the corresponding expression (3.53) with a minus sign in front of the mixed terms.

We add the two expressions and obtain

$$E_h^{n+1}:=\sum_{i=1}^{3}\left\{\left|\frac{1}{h}(\tilde{u}_1^{i,n+1}-\tilde{u}_0^{i,n+1})\right|^2+\left|\frac{1}{h}(\tilde{p}_1^{i,n+1}-\tilde{p}_0^{i,n+1})\right|^2+\beta^2|\tilde{p}_0^{i,n+1}|^2+\beta^2|\tilde{u}_0^{i,n+1}|^2\right\}$$

$$=E_h^n+2\beta\frac{1}{h}\sum_{i=1}^{3}[\tilde{p}_0^{i,n}\tilde{u}_1^{i,n}-\tilde{u}_0^{i,n}\tilde{p}_1^{i,n}]+2\beta\frac{1}{h}\sum_{i=1}^{3}[\tilde{p}_0^{i,n+1}\tilde{u}_1^{i,n+1}-\tilde{u}_0^{i,n+1}\tilde{p}_1^{i,n+1}]. \tag{3.54}$$

We are going to express the mixed terms in (3.54) in terms of final and boundary data,

$$0=\int_0^T\sum_{j=1}^{N}\left(\ddot{\tilde{u}}_j^i-\frac{1}{h^2}(\tilde{u}_{j-1}^i-2\tilde{u}_j^i+\tilde{u}_{j+1}^i)\right)\tilde{p}_j^i\,\mathrm{d}t$$

$$=kh\sum_{j=0}^{N}|\dot{\tilde{u}}_j^i(T)|^2v_i^i+kh\sum_{j=0}^{N}|\tilde{u}_j^i(T)|^2w_j^i$$

$$-\frac{1}{h}\int_0^T(u_0^ip_1^i-p_0^iu_1^i)\,\mathrm{d}t+\frac{1}{h}\int_0^T(u_N^ip_{N+1}^i-u_{N+1}^ip_N^i)\,\mathrm{d}t.$$

That is

$$\frac{1}{h}\sum_{i=1}^{3}\int_0^T(\tilde{p}_0^{i,n}\tilde{u}_1^{i,n}-\tilde{u}_0^{i,n}\tilde{p}_1^{i,n})\,\mathrm{d}t=-k\sum_{j=1}^{N}k\{|\tilde{u}_j^{i,n}(T)|^2+|\dot{\tilde{u}}_j^{i,n}(T)|^2\}$$

$$-\frac{1}{vh^2}\sum_{i=1}^{3}\int_0^T|\tilde{p}_{N+1}^{i,n}-\tilde{p}_N^{i,n}|^2\,\mathrm{d}t. \tag{3.55}$$

Using (3.55) in (3.54) we arrive at the recursion

$$\int_0^T E_h^{n+1}\,\mathrm{d}t=\int_0^T E_h^n\,\mathrm{d}t-2\beta k\sum_{j=0}^{N}k[(|\tilde{u}_j^{i,n}(T)|^2+|\tilde{u}_j^{i,n+1}(T)|^2)w_j^i+(|\dot{\tilde{u}}_j^{i,n}(T)|^2+|\dot{\tilde{u}}_j^{i,n+1}(T)|^2)v_j^i]$$

$$-2\beta\frac{1}{v}\sum_{i=1}^{3}\int_0^T\left\{\left|\frac{\tilde{p}_{N+1}^{i,n}-\tilde{p}_N^{i,n}}{h}\right|^2+\left|\frac{\tilde{p}_{N+1}^{i,n+1}-\tilde{p}_N^{i,n+1}}{h}\right|^2\right\}\mathrm{d}t. \tag{3.56}$$

Note that if we multiply by h^2 on both sides of (3.56) and if we put $\beta = (\tilde{\beta}/h)$, E_h^n becomes independent of h. Indeed (3.44), (3.45) become independent of h.

Recursion (3.56) is the crucial part in the convergence proof. Iterating (3.56) down to $n = 1$, we see that

$$\sum_{j=1}^{N} (|\tilde{u}_j^{i,n}(T)|^2 w_j^i + |\dot{\tilde{u}}_j^{i,n}(T)|^2 v_j^i)\ h \to 0, \tag{3.57}$$

$$\int_0^T \left| \frac{\tilde{p}_{N+1}^{i,n} - \tilde{p}_N^{i,n}}{h} \right|^2 \mathrm{d}t \to 0, \quad n \to \infty. \tag{3.58}$$

$$\int_0^T E_h^n \,\mathrm{d}t < \infty \quad \forall n \geqslant 1. \tag{3.59}$$

Eq. (3.59) implies weak convergence in $L^2(0,T)$ of the error terms in (3.54) (first on a subsequence and then one shows actually convergence of the entire sequence). Eqs. (3.57), (3.58) imply that the weak limit is actually zero. Indeed, arguing as in [14], we obtain

Theorem 12. *The sequence* $(u_j^{i,n}, p_j^{i,n})_{\substack{j=0:N+1\\ i=1:3}}$, $n \in \mathbb{N}$ *converges in* $C(0,T;\mathbb{R}^{3N+6} \times \mathbb{R}^{3N+6})$ *to the solution* (u_j^i, p_j^i) *of the optimality system* (3.40)–(3.47) *with n deleted.*

4. Numerical simulation

As mentioned in the introduction, the case of out-of-plane displacements is rather special, and this restriction was made only for the sake of simplicity. We, therefore, decided to present an example where in-plane displacements are dealt with. Indeed, 3-d-networks have been studied too, but the results are more difficult to visualize in a paper. We take a tripod of three strings as in (1.1). The difference is that now the quantities are vectorial $u_i = u_{i1}e_i + u_{i2}e_i^{\perp}$, where e_i are the unit vectors along the edges i starting at the multiple node (see [9]). We take $l = 1$ for all strings and take $h = 1/20$, $k = 100$, $\beta = 400$. We discretized in time using the standard $(\frac{1}{2}, \frac{1}{4})$-Newmark scheme with $\Delta t = 0.025$. We iterated the domain decomposition scheme about 15–25 times. The plots of the network under control (Figs. 1 and 2) clearly show the effect of our optimal controls. As a measure of convergence, we display only the difference in the traces across the interfacial node (at the center of the network) and the norm of the final-states. The convergence can be considerably improved by taking into account relaxation between actual and remote iterates and by using a Gauß–Seidel-type iteration. After all, the convergence is linear.

The main reason for introducing the domain decomposition algorithm of the semi-discrete and, ultimately, the fully discrete model is, however, to use the inherent parallelism on a parallel computer, or a workstation cluster. In fact, the parallelisation can even be traced down to the element (cell) level, where the local optimal control problem corresponding to the local optimality system (3.41)–(3.46) (see [8] for the explanation on the continuous level) can be solved even analytically, so that the computation can be done on a massive parallel machine as in [2]. The parallelization which we have in mind and which has not yet been implemented, however, will be pursued on

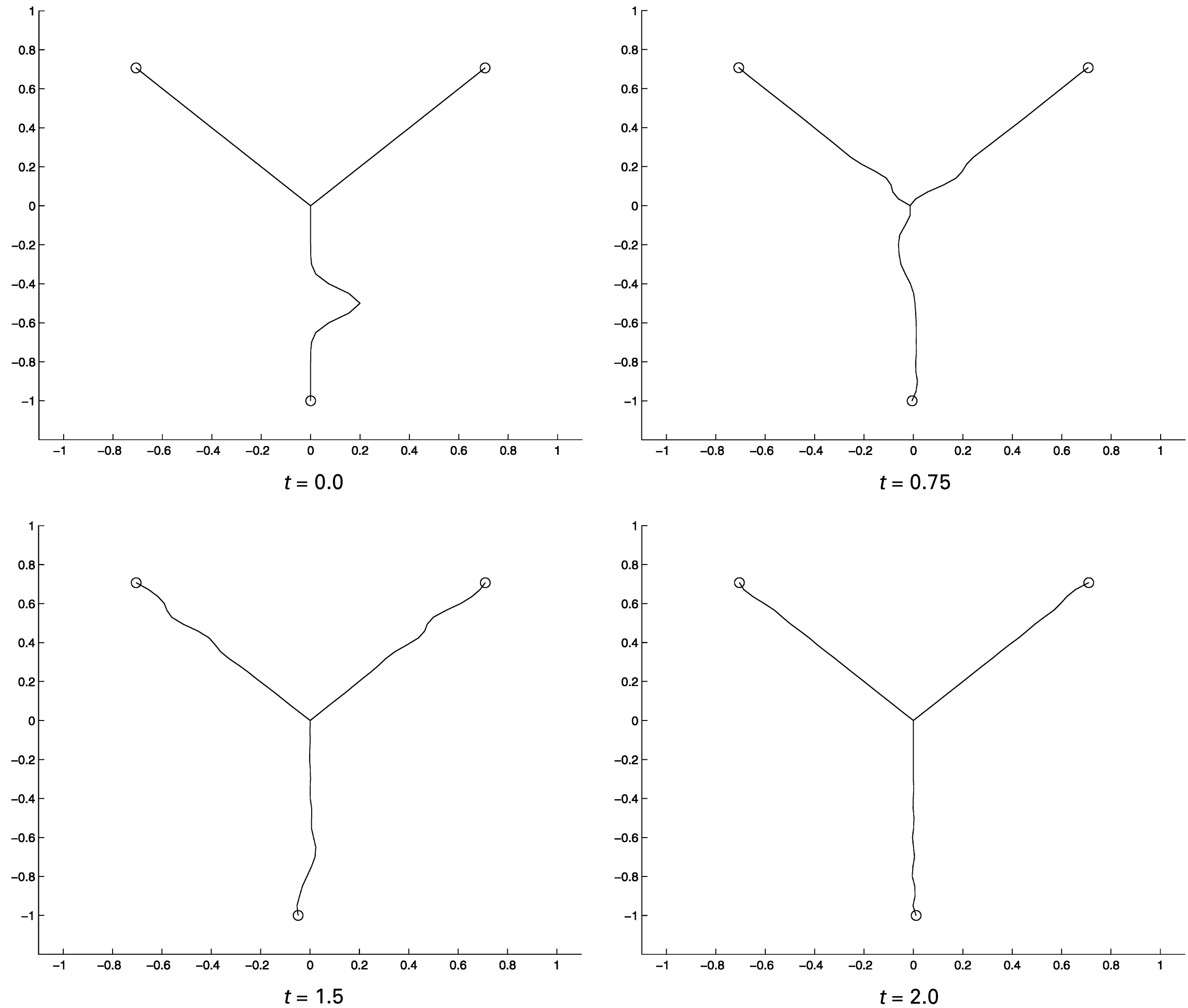

Fig. 1. A controlled network of elastic strings.

a medium grain level corresponding to the edges in the graph, i.e., to the substructures consisting of single strings (or beams). As we have, up to now, implemented only a serial version we confine ourselves with this short account of numerical experiments. A more detailed discussion of the numerical and implementational details is beyond the scope of this paper.

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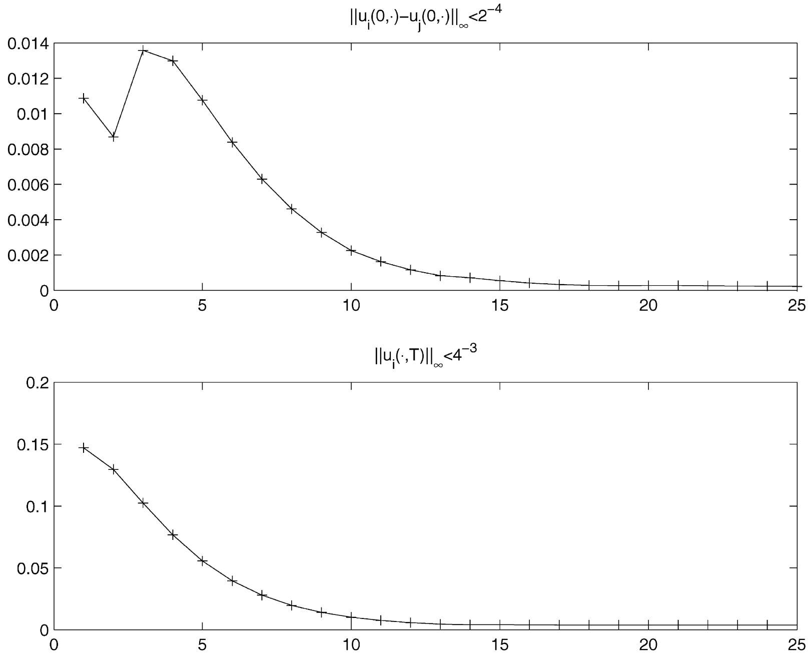

Fig. 2. Errorplots.

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